# Link invariant of the Izergin–Korepin model

P Pant and F Y Wu

Department of Physics and Center for Interdisciplinary Research on Complex Systems, Northeastern University, Boston, MA 02115, USA

**Abstract.** The link invariant associated with the Izergin–Korepin 19-vertex model is deduced using the method of statistical mechanics. It is shown that the Izergin–Korepin model leads to an invariant which is precisely the 3-state Akutsu–Wadati polynomial, previously known only for 2- and 3-braid knots. We give a table of the invariant for all knots and links up to seven crossings.

## 1. Introduction

Knots and links are planar projections of embeddings of circles in $\mathbb{R}^3$. Link invariants are algebraic quantities associated with the planar projections, which remain unchanged when the links are deformed. After the discovery of the Jones polynomial in 1985 [1], it is now well known that link invariants can be generated from exactly solvable models in statistical mechanics. For reviews of this development see [2–4].

Most link invariants, such as the Jones polynomials, can be computed recursively for all knots and links with the help of a Skein relation, starting from that of an unknot. On the other hand, there exist invariants, such as the Akutsu–Wadati polynomials [5] and the invariant associated with the chiral Potts model [6], for which the known Skein relation does not conveniently yield all invariants. In such instances, the method of statistical mechanics becomes an alternate and viable way of generating invariants. Here, we evaluate the invariant associated with the Izergin–Korepin model [7], and show that it leads to precisely the 3-state Akutsu–Wadati polynomial, previously computed only for 2- and 3-braid knots. In the appendix we give a table of the invariant for all knots and links up to seven crossings.

## 2. Formulation

We first briefly outline elements of generating link invariants from soluble vertex models. For definiteness, we follow the notations of [3].

Starting from an oriented knot or link $K$, one first deforms lines of $K$ to form a piecewise-linear graph $\mathcal{L}$. The example of forming $\mathcal{L}$ for the link $5_1^2$ is shown in figure 1. One next defines a vertex model on $\mathcal{L}$, which consists of vertices of degree 2 and two types of vertices of degree 4, $+$ and $-$, shown in figure 2. The objective is to construct vertex weights such that the resulting partition function of the vertex model is an invariant.

Consider generally an $N$-state vertex model on $\mathcal{L}$ for which each edge can be in $N$ distinct states. We denote the edge states by indices $\{a, b, c, \ldots\} \in \mathcal{J}$, where $\mathcal{J}$ is a set
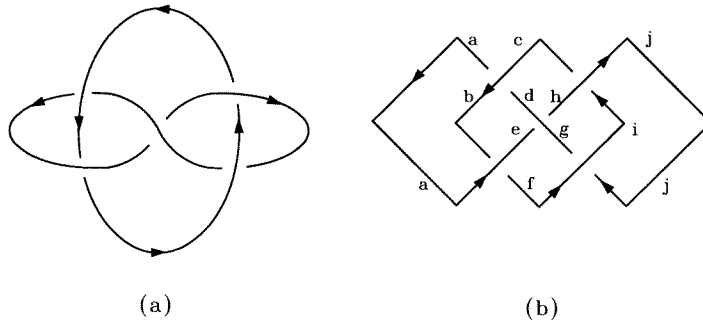
**Figure 1.** (*a*) The oriented link $5_1^2$. (*b*) The associated piecewise-linear graph $\mathcal{L}$.
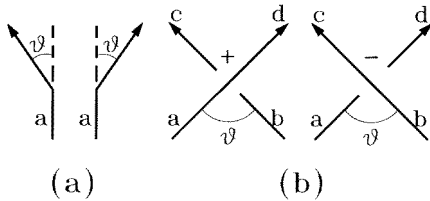


**Figure 2.** (*a*) Vertices of degree 2. (*b*) Vertices of degree 4.

of $N$ numerical values. To each vertex of degree 2 whose edge state is $a$ as shown in figure 2(*a*), associate a weight

$$\omega^*(a) = t^{a\theta/2\pi} \qquad \text{if the line turns an angle } \theta \text{ to the left}$$
$$= t^{-a\theta/2\pi} \qquad \text{if the line turns an angle } \theta \text{ to the right} \tag{1}$$

where $t$ is a factor at our disposal. To each vertex of degree 4 whose four incident edges are in the respective states $a, b, c, d$ as shown in figure 2(*b*), associate an 'enhanced' vertex weight

$$\omega_\pm^*(a, d|b, c) = t^{(a+c-b-d)\theta/4\pi} \omega_\pm(a, d|b, c). \tag{2}$$

Here, the $\pm$ signs refer to the two types $\pm$ of vertices, $\omega_\pm$ are vertex weights to be deduced as explained below, and $\theta$ is the angle formed by the two incoming arrows. Then, provided that the Reidemeister conditions depicted in figure 3 are satisfied, the partition function of the vertex model on $\mathcal{L}$,

$$Z_{\text{vertex}}(\omega^*) = \sum_{\text{edge states}} \prod \omega_\pm^*(a, d|b, c) \prod \omega^*(a) \tag{3}$$

where the first product is over all vertices of degree 4 and the second product over all vertices of degree 2, is an invariant of the knot $K$. Of course, one needs to ascertain that the invariant so obtained is unique, namely, it is independent of the angle $\theta$ chosen. Indeed, it has been shown [8] that the invariant so obtained does possess this invariance property, provided that the weights $\omega_\pm$ are charge-conserving, namely, satisfying

$$\omega_\pm(a, d|b, c) = 0 \qquad \text{unless } a + b = c + d. \tag{4}$$

For charge-conserving models, the enhanced weights reduce to

$$\omega_\pm^*(a, d|b, c) = t^{(a-d)\theta/2\pi} \omega_\pm(a, d|b, c) \tag{5}$$
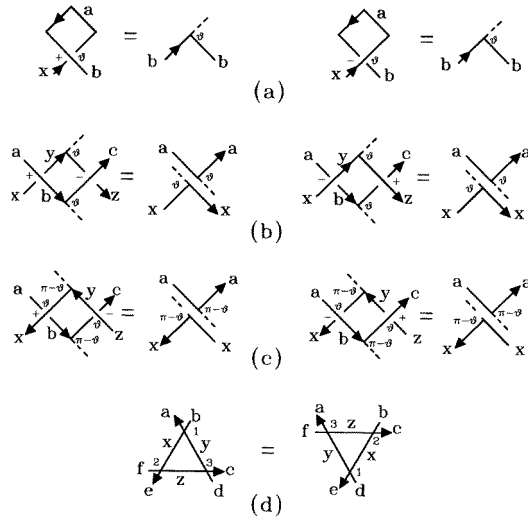
**Figure 3.** Reidemeister moves.

and, after introducing (5), the conditions depicted in figure 3 can be written as

$$\sum_{a \in \mathcal{J}} t^a \omega_{\pm}(a, b | x, a) = \delta_{bx} \tag{6a}$$

$$\sum_{b,y \in \mathcal{J}} \omega_{\pm}(a, b | x, y) \omega_{\mp}(y, z | b, c) = \delta_{ac} \delta_{xz} \tag{6b}$$

$$\sum_{b,y \in \mathcal{J}} t^{b-a} \omega_{\pm}(y, x | a, b) \omega_{\mp}(b, c | z, y) = \delta_{ac} \delta_{xz} \tag{6c}$$

$$\sum_{x,y,z \in \mathcal{J}} t^{(x+y+z-a-c-e)\theta/2\pi} \omega_1(y, a | b, x) \omega_2(x, e | f, z) \omega_3(z, c | d, y)$$

$$= \sum_{x,y,z \in \mathcal{J}} t^{(b+d+f-x-y-z)\theta/2\pi} \omega_1(d, y | x, e) \omega_2(b, x | z, c) \omega_3(f, z | y, a). \tag{6d}$$

Here, all summations are taken over the set $\mathcal{J}$ of edge states, and the indices $\{1, 2, 3\}$ in (6d) stand for the six equations with the replacements

$$\{1, 2, 3\} \rightarrow \{+--\}, \{+, -, +\}, \{+, +, -\}, \{-, +, -\}, \{-, -, +\}, \{-, +, +\}. \tag{7}$$

Finally, with weights satisfying (6a)–(6d), one writes the invariant for the knot $K$ as

$$I_K = Z_{\text{vertex}}(\omega^*) / Z_{\text{unknot}}(\omega^*) \tag{8}$$

so that it is normalized to $I_{\text{unknot}} = 1$, where

$$Z_{\text{unknot}}(\omega^*) = \sum_{a \in \mathcal{J}} t^a \tag{9}$$

is the partition function of the vertex model associated with an unknot.

The crux of matter in this formulation is the construction of the weights $\omega_{\pm}$. Here, the soluble model comes into play. It has been established [2–4] that if one takes

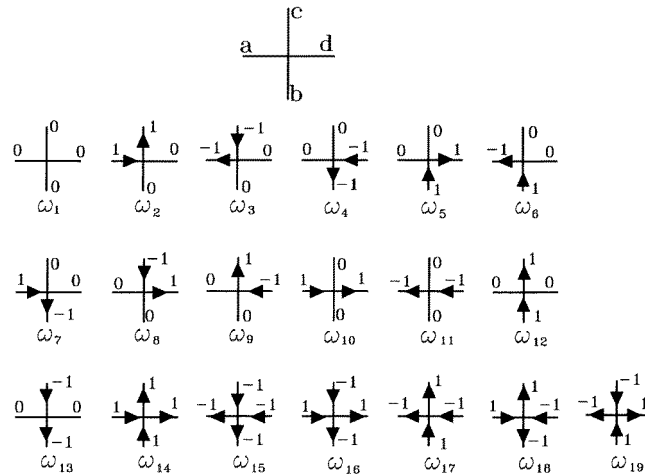$$\omega_{\pm}(a, d | b, c) \sim \lim_{u \to \pm \infty} \omega(a, d | b, c | u) \tag{10}$$

**Figure 4.** Vertex configurations of the 19-vertex model.

where $\omega(a, d|b, c|u)$ is the solution of the Yang–Baxter equation,

$$\sum_{x,y,z\in\mathcal{J}} \omega(x, b|y, a|u - w)\omega(f, z|e, x|v - u)\omega(z, c|d, y|v - w)$$

$$= \sum_{x,y,z\in\mathcal{J}} \omega(e, x|d, y|u - w)\omega(z, c|x, b|v - u)\omega(f, z|y, a|v - w) \tag{11}$$

of a vertex model, and $u$ is a rapidity factor appearing naturally in the solution, then the six conditions (6*d*) are automatically satisfied. It therefore remains only to satisfy the remaining conditions (6*a*)–(6*c*), which is a much easier task.

## 3. The Izergin–Korepin model

We now apply the above formulation to the Izergin–Korepin model [7].

The Izergin–Korepin model is a charge-conserving 19-vertex model on the square lattice. Let the edges of a square lattice be either oriented or unoriented so that each edge can be in $N = 3$ states. We further require that there are always the same number of incoming and outgoing arrows at each vertex. Then, there are 19 allowed vertex configurations which are shown in figure 4, and this gives rise to a 19-vertex model. Izergin and Korepin [7] found the following solution of the Yang–Baxter equation which, in the notations of [9], reads

$$\omega_1 = \omega(0, 0|0, 0|u) = e^u p^3 - e^{-u} p^{-3} + p^{-3} - p^3 + p^{-1} - p + p^5 - p^{-5}$$

$$\omega_2 = \omega_4 = \omega(1, 0, |0, 1|u) = \omega(0, -1| - 1, 0|u) = e^{-u}(p^{-1} - p^{-5}) + p^5 - p$$

$$\omega_3 = \omega_5 = \omega(-1, 0|0, -1|u) = \omega(0, 1|1, 0|u) = e^u(p^5 - p) + p^{-1} - p^{-5}$$

$$\omega_6 = \omega_8 = \omega(-1, 0|1, 0|u) = \omega(0, 1|0, -1|u) = e^u(p^4 - 1) + 1 - p^4$$

$$\omega_7 = \omega_9 = \omega(1, 0| - 1, 0|u) = \omega(0, -1|0, 1|u) = e^{-u}(1 - p^{-4}) + p^{-4} - 1$$

$$\omega_{10} = \omega_{11} = \omega_{12} = \omega_{13} = \omega(1, 1|0, 0|u) = \omega(-1, -1|0, 0|u)$$

$$= \omega(0, 0|1, 1|u) = \omega(0, 0| - 1, -1|u) = e^u p^3 - e^{-u} p^{-3} + p^{-3} - p^3$$

$$\omega_{14} = \omega_{15} = \omega(1, 1|1, 1|u) = \omega(-1, -1| - 1, -1|u) = e^u p^5 - e^{-u} p^{-5} + p^{-1} - p$$

$$\omega_{16} = \omega_{17} = \omega(1, 1| - 1, -1|u) = \omega(-1, -1|1, 1|u) = e^u p - e^{-u} p^{-1} + p^{-1} - p$$

$$\omega_{18} = \omega(1, -1| - 1, 1|u) = e^{-u}(p^{-1} - p^{-5} - p + p^{-3}) + p^5 - p^{-3}$$

$$\omega_{19} = \omega(-1, 1|1, -1|u) = e^{u}(p^5 - p^3 - p + p^{-1}) + p^3 - p^{-5}. \tag{12}$$

Here, the three edge states are labelled by $\mathcal{J} = \{-1, 0, +1\}$ with $+1$ $(-1)$ representing an arrow pointing towards right or upward (left or downward) and $0$ denoting no arrows.

Next we write

$$\omega_{\pm}(a, d|b, c) = A_{\pm} \lim_{u \to \pm\infty} \omega(a, d|b, c|u)e^{-|u|} \tag{13}$$

where $A_{\pm}$ are constants. It is then a relatively simple matter to verify that conditions (6a)–(6c) are all satisfied provided that one takes

$$p = t^{-1/2} \qquad A_{\pm} = \pm t^{\pm 7/2}. \tag{14}$$

Thus, using (13) and (14), we find from (12)

$$\omega_{\pm}(0, 0|0, 0) = t^{\pm 2}$$

$$\omega_{-}(1, 0|0, 1) = \omega_{-}(0, -1| - 1, 0) = t^{-1} - t^{-3}$$

$$\omega_{+}(1, 0|0, 1) = \omega_{+}(0, -1| - 1, 0) = 0$$

$$\omega_{+}(-1, 0|0, -1) = \omega_{+}(0, 1|1, 0) = t - t^3$$

$$\omega_{-}(-1, 0|0, -1) = \omega_{-}(0, 1|1, 0) = 0$$

$$\omega_{+}(-1, 0|1, 0) = \omega_{+}(0, 1|0, -1) = t^{3/2} - t^{7/2}$$

$$\omega_{-}(-1, 0|1, 0) = \omega_{-}(0, 1|0, -1) = 0$$

$$\omega_{-}(1, 0| - 1, 0) = \omega_{-}(0, -1|0, 1) = t^{-3/2} - t^{-7/2}$$

$$\omega_{+}(1, 0| - 1, 0) = \omega_{+}(0, -1|0, 1) = 0$$

$$\omega_{\pm}(1, 1|0, 0) = \omega_{\pm}(-1, -1|0, 0) = \omega_{\pm}(0, 0|1, 1) = \omega_{\pm}(0, 0| - 1, -1) = t^{\pm 2}$$

$$\omega_{\pm}(1, 1|1, 1) = \omega_{\pm}(-1, -1| - 1, -1) = t^{\pm 1}$$

$$\omega_{\pm}(1, 1| - 1, -1) = \omega_{\pm}(-1, -1|1, 1) = t^{\pm 3}$$

$$\omega_{-}(1, -1| - 1, 1) = t^{-1} - t^{-2} - t^{-3} + t^{-4}$$

$$\omega_{+}(1, -1| - 1, 1) = 0$$

$$\omega_{+}(-1, 1|1, -1) = t - t^2 - t^3 + t^4$$

$$\omega_{-}(-1, 1|1, -1) = 0. \tag{15}$$

## 4. Knot invariant

It is now straightforward to evaluate the invariant for any knot or link $K$ by substituting weights (15) into (3). For the link $K = 5_1^2$, for example, we construct $\mathcal{L}$ and label the edges as shown in figure 1(b), where all angles are taken to be $\pi/2$. This leads to the partition function

$$Z_{5_1^2}(\omega^*) = \sum_{a\ldots j=0,\pm 1} t^{(3a+b+c+f+i-3j)/4} \omega_{-}^*(d, a|c, b)\omega_{-}^*(b, f|a, e)$$

$$\times \omega_{-}^*(e, h|g, d)\omega_{+}^*(h, j|i, c)\omega_{+}^*(f, i|j, g)$$

$$= \sum_{a\ldots j=0,\pm 1} t^{(2a+2b+c+d+e+f-4j)/4} \omega_{-}(d, b|c, a)\omega_{-}(b, f|a, e)$$

$$\times \omega_{-}(e, h|g, d)\omega_{+}(h, j|i, c)\omega_{+}(f, i|j, g). \tag{16}$$
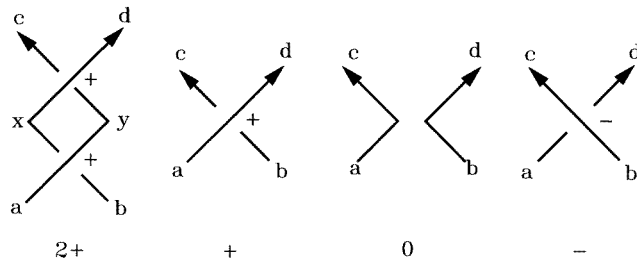
**Figure 5.** Line configurations for the Skein relation.

Furthermore, the partition function (9) of an unknot is computed as

$$Z_{\text{unknot}}(\omega^*) = \sum_{a=0,\pm1} t^a = 1 + t + t^{-1}. \tag{17}$$

After some algebra, we obtain from (8) the invariant

$$I_{5_1^2} = t^{-10}(1 - 2t - t^2 + 4t^3 - 3t^4 - t^5 + 5t^6 - 3t^7 - t^8 + 5t^9$$
$$-2t^{10} - t^{11} + 3t^{12} - t^{13} - t^{14} + t^{15}).$$

Generally, the partition sums such as the one in (16) can be performed using a symbolic computation package, after first eliminating summation variables using the charge-conserving condition (4). Results of our computation are given in the appendix.

*4.1. The Skein relation*

It is readily verified that weights (15) satisfy

$$\sum_{x,y=\{0,\pm1\}} \omega_+(a, y|b, x)\omega_+(x, d|y, c) = (t - t^3 + t^4)\omega_+(a, d|b, c)$$
$$+(t^4 - t^5 + t^7)\delta_{ac}\delta_{bd} - t^8\omega_-(a, d|b, c) \qquad \text{for all } a, b, c, d \tag{18}$$

an identity in which the weights $\omega_\pm$ are represented graphically in figure 5, where 2+ denotes the twist of two consecutive + crossings as shown. This identity then implies the Skein relation

$$I_{2+}(t) = (t - t^3 + t^4)I_+(t) + (t^4 - t^5 + t^7)I_0(t) - t^8 I_-(t) \tag{19}$$

where $I_{2+}, I_+, I_0, I_-$ are invariants of four knots which differ only in the region of a small disk represented by the configurations 2+, +, 0, − of figure 5, respectively. We find that the Skein relation (19) is identical to that of the 3-state Akutsu–Wadati polynomial [10]. The 3-state Akutsu–Wadati polynomial is a knot invariant computed [5] from the Skein relation (19) obtained from the Zamolodchikov–Fateev model [11]. However, the computation was restricted to knots of 2- and 3-braids and links of 2-braids, and does not include other more general links and knots such as the knot $6_1$. While it has since been pointed out [4] that the Akutsu–Wadati polynomial can also be computed in a fashion similar to ours, such a program has not been carried out. Indeed, we have independently verified that, by following our formulation, the Zamolodchikov–Fateev model yields the same $\omega_\pm$ as that given in (15). This identifies that the invariant obtained from the Izergin–Korepin model is precisely the 3-state Akutsu–Wadati polynomial.

*4.2. The mirror image of a knot*

The mirror image $K^*$ of a knot $K$ is obtained by reflecting $K$ about a line, say, the $y$-axis. This process interchanges the $\pm$ crossings and the relative arrow positionings at a cross. For a $\pm$ crossing in $K$ with the weight $\omega_\pm(a, d|b, c)$ (cf figure 2(b)), the image in $K^*$ is a $\mp$ crossing with the weight $\omega_\mp(b, c|a, d)$. Perusal of (15) shows that, for all $a, b, c, d$, these two weights are related by the simple replacement of $t \to t^{-1}$. Furthermore, since a right-turn in $K$ becomes a left-turn in $K^*$, and vice versa, the weight (1) is also related by the replacement $t \to t^{-1}$. It follows that the invariant for the mirror image $K^*$ is simply

$$I_{K^*}(t) = I_K(t^{-1}). \tag{20}$$

**Acknowledgment**

**Appendix. Table of invariants**

In this appendix we give invariant $I_K(t)$ deduced from the Izergin–Korepin 19-vertex model for all knots and links up to seven crossings. We adopt the shorthand notation of

$$I_K(t) = t^m(a_0 + a_1 t - a_2 t^2 - a_3 t^3 \ldots) \to [m][a_0, a_1, \bar{a}_2, \bar{a}_3 \ldots] \tag{A1}$$

where $\{a_i\}$, $i = 0, 1, 2, \ldots$ are nonnegative integers, and a bar denotes that the coefficient is negative. For example, the invariant for the knot $6_1$ is

$$t^{-12}(1 - t - t^2 + 2t^3 - t^4 - 2t^5 + 3t^6 - 3t^8 + 4t^9 - 4t^{11} + 4t^{12} - 3t^{14} + 2t^{15} - t^{17} + t^{18})$$
$$\to [-12][1, \bar{1}, \bar{1}, 2, \bar{1}, \bar{2}, 3, 0, \bar{3}, 4, 0, \bar{4}, 4, 0, \bar{3}, 2, 0, \bar{1}, 1]. \tag{A2}$$

For links we also list the writhe $w(K) = n_+ - n_-$, where $n_\pm$ is the number of $\pm$ crossings. Invariants for links and the knots $6_1, 7_2, 7_4, 7_6, 7_7$ are new, which were not reported in [5].

| $K$ | $I_K(t)$ |
|---|---|
| $3_1$ | $[2][1, 0, 0, 1, 0, \bar{1}, 1, \bar{1}, \bar{1}, 1]$ |
| $4_1$ | $[-6][1, \bar{1}, \bar{1}, 2, \bar{1}, \bar{1}, 3, \bar{1}, \bar{1}, 2, \bar{1}, \bar{1}, 1]$ |
| $5_1$ | $[4][1, 0, 0, 1, 0, \bar{1}, 1, 0, \bar{1}, 1, 0, \bar{2}, 1, 0, \bar{1}, 1]$ |
| $5_2$ | $[2][1, \bar{1}, 0, 3, \bar{2}, \bar{1}, 4, \bar{3}, \bar{1}, 3, \bar{2}, \bar{1}, 2, \bar{1}, \bar{1}, 1]$ |
| $6_1$ | $[-12][1, \bar{1}, \bar{1}, 2, \bar{1}, \bar{2}, 3, 0, \bar{3}, 4, 0, \bar{4}, 4, 0, \bar{3}, 2, 0, \bar{1}, 1]$ |
| $6_2$ | $[-4][1, \bar{1}, \bar{1}, 3, \bar{1}, \bar{3}, 5, \bar{1}, \bar{5}, 6, 0, \bar{6}, 6, 0, \bar{5}, 4, 0, \bar{2}, 1]$ |
| $6_3$ | $[-9][1, \bar{2}, \bar{1}, 5, \bar{4}, \bar{3}, 9, \bar{5}, \bar{5}, 11, \bar{5}, \bar{5}, 9, \bar{3}, \bar{4}, 5, \bar{1}, \bar{2}, 1]$ |
| $2_1^2$ | $[1][1, 0, 0, 1, 0, 0, 1], w(K) = -2$ |
| $4_1^2$ | $[3][1, 0, 0, 1, 0, \bar{1}, 1, 0, \bar{1}, 2, 0, \bar{1}, 1], w(K) = -4$ |
| | $[19][1, 0, 0, 1, 0, \bar{1}, 1, 0, \bar{1}, 2, 0, \bar{1}, 1], w(K) = 4$ |
| $5_1^2$ | $[-10][1, \bar{2}, \bar{1}, 4, \bar{3}, \bar{1}, 5, \bar{3}, \bar{1}, 5, \bar{2}, \bar{1}, 3, \bar{1}, \bar{1}, 1], w(K) = -1$ |
| $6_1^2$ | $[5][1, 0, 0, 1, 0, \bar{1}, 1, 0, \bar{1}, 1, 0, \bar{1}, 1, 1, \bar{1}, 1, 0, \bar{1}, 1], w(K) = -6$ |
| | $[29][1, 0, 0, 1, 0, \bar{1}, 1, 0, \bar{1}, 1, 0, \bar{1}, 1, 1, \bar{1}, 1, 0, \bar{1}, 1], w(K) = 6$ |
| $6_2^2$ | $[3][1, \bar{1}, 0, 3, \bar{2}, \bar{2}, 5, \bar{3}, \bar{3}, 7, \bar{3}, \bar{3}, 5, \bar{2}, \bar{2}, 3, 0, \bar{1}, 1], w(K) = 6$ |
| $6_3^2$ | $[-21][1, \bar{2}, \bar{1}, 5, \bar{3}, \bar{3}, 8, \bar{4}, \bar{5}, 9, \bar{4}, \bar{5}, 8, \bar{2}, \bar{3}, 4, 0, \bar{1}, 1], w(K) = -6$ |

$\quad\quad [-5][1, \bar{2}, \bar{1}, 5, \bar{3}, \bar{3}, 8, \bar{4}, \bar{5}, 9, \bar{4}, \bar{5}, 8, \bar{2}, \bar{3}, 4, 0, \bar{1}, 1], w(K) = 2$

$6_1^3 \quad [-20][1, \bar{1}, \bar{1}, 4, \bar{1}, \bar{3}, 7, \bar{5}, 8, 1, \bar{7}, 7, 0, \bar{6}, 5, 1, \bar{2}, 1], w(K) = -6$

$\quad\quad [-4][1, \bar{1}, \bar{1}, 4, \bar{1}, \bar{3}, 7, \bar{5}, 8, 1, \bar{7}, 7, 0, \bar{6}, 5, 1, \bar{2}, 1], w(K) = 2$

$6_2^3 \quad [-9][1, \bar{3}, \bar{1}, 8, \bar{6}, \bar{4}, 14, \bar{7}, \bar{6}, 17, \bar{6}, \bar{7}, 14, \bar{4}, \bar{6}, 8, \bar{1}, \bar{3}, 1]w(K) = 0$

$6_3^3 \quad [-12][1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 2, 1], w(K) = -2$

$\quad\quad [4][1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 2, 1], w(K) = 6$

$7_1 \quad [6][1, 0, 0, 1, 0, \bar{1}, 1, 0, \bar{1}, 1, 0, \bar{1}, 1, 0, \bar{1}, 0, 0, \bar{1}, 1, 0, \bar{1}, 1]$

$7_2 \quad [-23][1, \bar{1}, \bar{1}, 2, \bar{1}, \bar{2}, 3, 0, \bar{3}, 3, 0, \bar{4}, 3, 1, \bar{4}, 3, 1, \bar{2}, 2, 0, \bar{1}, 1]$

$7_3 \quad [4][1, \bar{1}, 0, 3, \bar{2}, \bar{2}, 5, \bar{2}, \bar{4}, 7, \bar{2}, \bar{6}, 8, \bar{2}, \bar{5}, 5, \bar{1}, \bar{3}, 2, 0, \bar{1}, 1]$

$7_4 \quad [2][1, \bar{2}, 1, 4, \bar{6}, 2, 6, \bar{9}, 3, 7, \bar{8}, 1, 7, \bar{7}, \bar{1}, 5, \bar{4}, \bar{1}, 3, \bar{1}, \bar{1}, 1]$

$7_5 \quad [4][1, \bar{1}, 0, 4, \bar{3}, \bar{3}, 9, \bar{5}, \bar{7}, 13, \bar{5}, \overline{10}, 14, \bar{4}, \bar{9}, 11, \bar{2}, \bar{6}, 5, 0, \bar{2}, 1]$

$7_6 \quad [-17][1, \bar{2}, 0, 5, \bar{7}, \bar{1}, 12, \overline{12}, \bar{3}, 17, \overline{13}, \bar{4}, 16, \overline{10}, \bar{5}, 12, \bar{5}, \bar{4}, 6, \bar{1}, \bar{2}, 1]$

$7_7 \quad [-9][1, \bar{3}, 0, 8, \bar{9}, \bar{2}, 16, \overline{14}, \bar{5}, 21, 15, \bar{7}, 20, \overline{11}, \bar{7}, 14, \bar{5}, \bar{5}, 6, \bar{1}, \bar{2}, 1]$

$7_1^2 \quad [-18][1, \bar{2}, 0, 3, \bar{5}, 2, 5, \bar{7}, 2, 6, \bar{7}, 1, 6, \bar{5}, 0, 5, \bar{3}, \bar{1}, 3, \bar{1}, \bar{1}, 1], w(K) = -3$

$\quad\quad [-10][1, \bar{2}, 0, 3, \bar{5}, 2, 5, \bar{7}, 2, 6, \bar{7}, 1, 6, \bar{5}, 0, 5, \bar{3}, \bar{1}, 3, \bar{1}, \bar{1}, 1], w(K) = 1$

$7_2^2 \quad [-16][1, \bar{2}, \bar{1}, 5, \bar{4}, \bar{4}, 10, \bar{4}, \bar{8}, 15, \bar{4}, \overline{11}, 15, \bar{3}, \overline{10}, 11, \bar{1}, \bar{6}, 5, 0, \bar{2}, 1], w(K) = -3$

$\quad\quad [-8][1, \bar{2}, \bar{1}, 5, \bar{4}, \bar{4}, 10, \bar{4}, \bar{8}, 15, \bar{4}, \overline{11}, 15, \bar{3}, \overline{10}, 11, \bar{1}, \bar{6}, 5, 0, \bar{2}, 1], w(K) = 1$

$7_3^2 \quad [-16][1, \bar{2}, \bar{1}, 5, \bar{3}, \bar{4}, 8, \bar{3}, \bar{8}, 11, \bar{2}, \bar{9}, 12, \bar{1}, \bar{8}, 8, 0, \bar{4}, 3, 0, \bar{1}, 1], w(K) = -3$

$7_4^2 \quad [-3][1, \bar{1}, \bar{1}, 4, \bar{1}, \bar{4}, 7, 0, \bar{7}, 8, 2, \overline{10}, 7, 3, \overline{10}, 6, 3, \bar{7}, 3, 1, \bar{2}, 1], w(K) = 3$

$7_5^2 \quad [-24][1, \bar{2}, \bar{1}, 6, \bar{5}, \bar{5}, 13, \bar{6}, \overline{11}, 18, \bar{5}, \overline{14}, 19, \bar{3}, \overline{13}, 14, \bar{1}, \bar{8}, 6, 1, \bar{2}, 1], w(K) = -7$

$\quad\quad [-8][1, \bar{2}, \bar{1}, 6, \bar{5}, \bar{5}, 13, \bar{6}, \overline{11}, 18, \bar{5}, \overline{14}, 19, \bar{3}, \overline{13}, 14, \bar{1}, \bar{8}, 6, 1, \bar{2}, 1], w(K) = 1$

$7_6^2 \quad [-13][1, \bar{3}, 1, 8, \overline{12}, \bar{1}, 19, \overline{19}, \bar{4}, 26, \overline{20}, \bar{7}, 26, \overline{15}, \bar{8}, 19, \bar{7}, \bar{7}, 9, \bar{1}, \bar{3}, 1], w(K) = -1$

$7_7^2 \quad [-11][1, 0, 0, 1, 0, 0, 1], w(K) = -1$

$\quad\quad [5][1, 0, 0, 1, 0, 0, 1], w(K) = 7$

$7_8^2 \quad [-16][1, \bar{1}, \bar{1}, 2, \bar{1}, \bar{2}, 2, 0, \bar{3}, 3, 1, \bar{3}, 3, 1, \bar{2}, 2, 1], w(K) = -3$

$7_1^3 \quad [-12][1, \bar{1}, 0, 5, \bar{4}, \bar{3}, 13, \bar{7}, \bar{9}, 19, \bar{7}, \overline{12}, 21, \bar{5}, \overline{12}, 16, \bar{3}, \bar{9}, 8, 0, \bar{3}, 1], w(K) = -1$

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